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THE

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AND

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OF THE

INSTITUTE OF ACTUARIES.

On a Method of finding the Product of two Factors by means of the Addition and Subtraction of Natural Numbers. By Samuel Linn Laundy, Esq., *Actuary to the London Mutual Life and Guarantee Society.*

[Read before the Institute, 31st December, 1855, and ordered by the Council to be printed.]

THE paper which I have the honour to submit is so elementary in its details that I have to claim the indulgence of the meeting; but, as the practical application of the simple formulæ to which I shall have occasion to refer may be interesting, I trust I may not unworthily occupy a short portion of your time.

A few months ago it occurred to me that tables to facilitate the process of multiplication, based upon the identity

$$ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2,$$

would be of extensive use in actuarial and other calculations, and would effect a considerable saving of labour—to an extent even greater than that acquired by the use of logarithms. This led me to give attention to the subject; and after a trial on a small scale, I deemed it of sufficient importance to construct for my own use a series of tables of the fourth parts of the squares of all integer numbers up to 100,000.

I may observe, the design of the tables I have so constructed is to abridge the labour attendant upon the usual method of finding the products of two given factors by the process of multiplication. In cases where factors are each composed of several digits, the ordinary operation of multiplication obviously involves considerable toil and liability to error. With a view, therefore, to lessen such labour, and as far as possible to ensure accuracy, I was induced to frame the tables I shall presently describe; and which will enable a computer, I think, to arrive at results almost without any mental effort beyond the most ordinary care.

The value and importance of tables for finding the products of numbers was brought under the notice of the members of this Institute by an able contributor to its *Journal,* in a paper entitled "On multiplication by a table of single entry,"* in which a remark is introduced as occurring somewhere in Gergonne's *Annales,* to the effect that "a table of single entry, sufficiently extensive, will serve to give the product of any two numbers by the aid of the processes of addition and subtraction ALONE, just as is the case in logarithmic computation." The writer then proceeds to explain how, by means of a table of squares, the product of any two given factors may be obtained. He says—"Suppose now that a and b are any two numbers, both even or both odd, we may proceed as follows: Take the two numbers which are the respective *half* sum and *half* difference of a and b, find the squares corresponding to these respectively from the given table; the difference of these squares will be the product of the given numbers a and b, as is apparent from the identity

$$ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2."$$

He then points out that when one of the given numbers is even and the other odd, either of the two given numbers may be diminished by a unit; and the given couple of numbers thus modified are then to be treated as before stated, and the result increased by the remaining one of the given numbers, viz., the one diminished by a unit.

In order, however, to obtain "the product of any two numbers by the aid of the processes of addition and subtraction *alone,*" other aid than that afforded by a table of squares must be at hand: for before entering a table of squares, the computer is under the necessity of *halving both the sum and difference* of the given factors; or, after entering the table for the squares of the given factors, he has to take the fourth part of such squares.

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* By J. J. Sylvester, M.A., F.R.S. (*Assurance Mag.*, vol. iv., p. 236.)

The tables now under notice, and by which it is proposed to gain the advantage above pointed out, contain the squares of $\left(\frac{\text{numbers}}{2}\right)=\left(\frac{\text{the squares of numbers}}{4}\right)$, and will not only render unnecessary the halving of the sum and difference, but will give *directly,* by single entry, and without any modification for odd numbers (as is requisite when a table of squares is used), the product of any two factors within the limits before mentioned.

The number to which the tables have been carried will render them valuable in many computations occurring in actuarial practice; and independently of the advantage of tables of this form for accomplishing the object proposed, over a table of squares, they will assist a computer to an extent five times greater than Barlow's Tables of Squares, and nearly twice as great as Hutton's.

Before proceeding to explain the use of the tables, I may be allowed to refer to the process by which they have been computed. To form each value separately by multiplication would require an amount of labour that few would voluntarily undergo: but considering the fact that the squares of numbers beginning from unity may be found by the successive addition of a constantly and uniformly increasing difference, it is evident that every tabular number may be found from that preceding it by the process of addition. For example:—

Expanding the binomial $\left(\frac{a+b}{2}\right)^2$, we obtain $\frac{a^2}{4}+\frac{ab}{2}+\frac{b^2}{4}$; and making $a=1$, this becomes $\frac{1}{4}+\frac{b}{2}+\frac{b^2}{4}$; and substituting all values from 0 upwards for b, we obtain the following series—

$$\begin{array}{llllllll|l}
 & & & & & & & & \Delta \\
(1) = \frac{1}{4} & & & = & \frac{1}{4} & & & & -\ \frac{2}{4} \\
(2) = \frac{1}{4} & + \frac{1}{2} & + \frac{1}{4} & = \frac{4}{4} & = 1 & = (1) & + \frac{3}{4} & & -\ \frac{2}{4} \\
(3) = \frac{1}{4} & + \frac{2}{2} & + \frac{4}{4} & = \frac{9}{4} & = 2\frac{1}{4} & = (2) & + \frac{5}{4} & & -\ \frac{2}{4} \\
(4) = \frac{1}{4} & + \frac{3}{2} & + \frac{9}{4} & = \frac{16}{4} & = 4 & = (3) & + \frac{7}{4} & & -\ \frac{2}{4} \\
(5) = \frac{1}{4} & + \frac{4}{2} & + \frac{16}{4} & = \frac{25}{4} & = 6\frac{1}{4} & = (4) & + \frac{9}{4} & & -\ \frac{2}{4} \\
(6) = \frac{1}{4} & + \frac{5}{2} & + \frac{25}{4} & = \frac{36}{4} & = 9 & = (5) & + \frac{16}{4} & & -\ \frac{2}{4}
\end{array}$$

Whence we find

$$\begin{array}{llllll}
(2) & = (1) & + & (1) & + & \Delta \\
(3) & = (2) & + & (1) & + & 2\Delta \\
(4) & = (3) & + & (1) & + & 3\Delta \\
(5) & = (4) & + & (1) & + & 4\Delta \\
(6) & = (5) & + & (1) & + & 5\Delta \\
(n) & = (n-1) & + & (1) & + & (n-1)\Delta
\end{array}$$

Applying this numerically, as in the following example, the tables up to a certain point, to which I shall presently refer, were calculated.

$a=1,\ b=0,\ 1,\ 2,\ 3,$ &c.

$\frac{a}{4}=\frac{1}{4}=25$

$\frac{b}{2}+\frac{b^2}{4}=$	$\frac{00}{25}$	·25	·25= (1)	Δ·50	994·75	990025·00=(1990)
Δ	50	·75	1·00= (2)	Δ·50	995·25	991020·25=(1991)
Δ	50	1·25	2·25= (3)	Δ·50	995·75	992016·00=(1992)
Δ	50	1·75	4·00= (4)	Δ·50	996·25	993012·25=(1993)
Δ	50	2·25	6·25= (5)	Δ·50	996·75	994009·00=(1994)
Δ	50	2·75	9·00= (6)	Δ·50	997·25	995006·25=(1995)
Δ	50	3·25	12·25= (7)	Δ·50	997·75	996004·00=(1996)
Δ	50	3·75	16·00= (8)	Δ·50	998·25	997002·25=(1997)
Δ	50	4·25	20·25= (9)	Δ·50	998·75	998001·00=(1998)
Δ	50	4·75	25·00=(10)	Δ·50	999·25	999000·25=(1999)

and proceeding in this manner, the whole of the subsequent numbers, to any assigned limit, may be found.

But the computation from No. (2000) and onward can be more conveniently continued by a modification of the above method. The numbers being tabulated to No. (1999), it is obvious from the circumstance of the digits 1 to 999 being common to every chiliad, and $\frac{b}{2}$ now occupying the place of thousands, that the three final digits will constantly recur. They may therefore be repeated to any desired length without the trouble of calculation, and the remaining digits supplied by inserting $\frac{b^2}{4}$ opposite the initial number of each table, and each succeeding number completed by filling in the digits of $\frac{b}{2}$ which stand in the place of thousands and upwards.

In pursuing the calculation—having regard to the labour of transcribing, and the risk of error attending it—I determined to arrange the three final figures of the values of (1) to (1999) in a tabular form, and have them printed. This being done, the work was continued as just described.

The following tabular example will show the arrangement adopted, and serve to illustrate the final process. The ordinary and thick black figures represent the portion printed; the old-style figures are those subsequently calculated and inserted in manuscript.

N. 50,000—50,499.

N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
5000	625 000 000	025 000	050 001	075 002	100 004	125 006	150 009	175 012	200 016	225 020
1	250 025	275 030	300 036	325 042	350 049	375 056	400 064	425 072	450 081	475 090
2	500 100	525 110	550 121	575 132	600 144	625 156	650 169	675 182	700 196	725 210
3	750 225	775 240	800 256	825 272	850 289	875 306	900 324	925 342	950 361	975 380
4	626 000 400	025 420	050 441	075 462	100 484	125 506	150 529	175 552	200 576	225 600
*	*	*	*	*	*	*	*	*	*	*
5045	636 300 625	325 850	351 076	376 302	401 529	426 756	451 984	477 212	502 441	527 670
6	552 900	578 130	603 361	628 592	653 824	679 056	704 289	729 522	754 756	779 990
7	805 225	830 460	855 696	880 932	906 169	931 406	956 644	981 882	*007 121	*032 360
8	637 057 600	082 840	108 081	133 322	158 564	183 806	209 049	234 292	259 536	284 780
9	310 025	335 270	360 516	385 762	411 009	436 256	461 504	486 752	512 001	537 250
N	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

A curious property will be observed to belong to these final digits, depending upon the properties of numbers in the decimal scale—viz., that for an even chiliad they occur in an ascending order, while those for an odd chiliad occur in a corresponding descending order; so that, as we proceed through the tables of the even chiliads, we obtain digits which recur in an inverted order in the odd chiliads. This property has been taken advantage of to test the accuracy of the three final digits throughout.

It may be proper to remark, that in each one of the columns headed with an odd digit the whole of the tabular numbers in such columns would naturally terminate with ·25, in addition to the numbers already entered therein. It was however deemed unnecessary to retain these decimals, from the circumstance that, whenever the sum of two numbers is odd, their difference will also be odd; and therefore the result (*i. e.*, the difference of the tabular numbers) would in no way be affected by such omission. Bearing this omission in mind, if it be desired to find the tabular value corresponding to any number within the Tables × 10, such value will be obtained by annexing 00 to the numbers in all the columns headed by an even digit, and 25 to the numbers in all the columns headed by an odd digit.

The arrangement adopted is similar to that of a table of logarithms, with this exception—that the final digit of any number whose tabular value is required must in every case be sought in the column headed by such final digit. Without this explanation a computer might err in the use of the column headed (0), or in seeking the tabular value of any number ending with a 0.

The constant digits at the left of the tabular numbers are not repeated, but arranged in the manner usual in logarithmic tables; and when the first of the final digits standing in any of the columns changes from a 9 to a 0 in any line, an asterisk is prefixed through the remaining part of the line, in order to catch the eye and direct attention to the change that will take place in the constants, which constants will in such case be found at the beginning of the next following line.

I will now proceed to show the practical application of the tables. Since $ab=\left(\frac{a+b}{2}\right)^2-\left(\frac{a-b}{2}\right)^2=\frac{(a+b)^2}{4}-\frac{(a-b)^2}{4}$, let $a+b=s$, $a-b=d$, and $T_s=\frac{s^2}{4}$, $T_d=\frac{d^2}{4}$; then $T_s-T_d=ab$. The interpretation of which is as follows:—To find the product of two given factors, from one fourth of the square of their sum take one fourth of the square of their difference; the remainder will be the product sought.

The tables I have been describing, being adapted to this formula, will therefore give directly the product of two factors by addition and subtraction alone. The following examples will illustrate the process:—

Required the product of ·53684 × 45·678.

$$\begin{array}{lrr} s. & 99362 & T_s = 2468201761 \\ d. & 8006 & T_d = \underline{16024009} \\ & & 24{\cdot}52177752 \end{array}$$

It is to be observed, that in finding the sum and difference of numbers the decimal point need not be regarded; in other words, the significant figures should for this purpose be written under each other as if they were integers.

In cases where either or both of the given factors consist of more than five digits, the tables will be still available by a slight modification of the last formula.* In such cases let the digits of the factor or factors be separated into two parts, each consisting of such a number of digits as may be most convenient; and, calling the left-hand digits of the one a', and those at its right a, then $a'10^n+a$ will represent the multiplicand (n being the number of digits to the right): similarly, the multiplier may be represented by $b'10^n+b$.

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* Since the above was written the author has received a valuable suggestion from Mr. Peter Gray, to insert in the tables the tabular differences and proportional parts, which will materially enhance the advantages afforded by them.

When the digits of one factor exceed five, and the other factor consists of five or any less number of digits, the formula becomes

$$(a'10^n+a)b=\left\{T_{s(a',b)}-T_{d(a',b)}\right\}10^n+\left\{T_{s(a,b)}-T_{d(a,b)}\right\}.$$

When the digits of both factors exceed five in number, the formula becomes

$$\frac{(a'10^n+\dot{a}).(b'10^n+b)}{4}=\frac{a'b'10^{2n}+(a'b+ab')10^n+ab}{4}=$$

$$\left\{T_{s(a',b')}-T_{d(a',b')}\right\}10^{2n}+\left\{T_{s(a',b)}-T_{d(a',b)}\right\}10^n+\left\{T_{s(a,b')}-T_{d(a,b')}\right\}10^n$$
$$+\left\{T_{s(a,b)}-T_{d(a,b)}\right\}.$$

Confining myself, however, to the illustration of the products of factors not consisting of more than six digits, I submit that the following rules, deduced from the two last preceding formulæ, will be found the most convenient in practice:—

1. To find the product of two factors, one consisting of six digits and the other of five or any less number of digits: Take the sum and difference of the first five digits of the greater number and the less number, and proceed as before. To the result so obtained annex a cypher; then add the product of the less number multiplied by the sixth digit of the greater number, and the sum of the two results will give the product sought.

Example: What is the product of 536846 × 45678?

s. 99362 T_s=2468201761
d. 8006 T_d= 16024009

24521777520
45678 × 6= 274068

24522051588 *Answer.*

2. To multiply two numbers each consisting of six digits: Take the sum and difference of the first five digits of each factor, and proceed as before. To the result annex two cyphers; to this add the product of the first five figures of the greater number, increased by a cypher, multiplied by the sixth digit of the less number; then add the product of the first five digits of the less number, increased by a cypher, multiplied by the sixth digit of the greater number; and, lastly, add the product of the two final digits.

Example: To find the product of 536847 × 456788.

$$\begin{array}{lr} s.\ 99362\quad T_s= & 2468201761 \\ d.\ 8006\quad T_d= & 16024009 \\ \hline & 245217775200 \\ {}^{*}536840\times 8= & 4294720 \\ {}^{*}456780\times 7= & 3197460 \\ {}^{*}7\times 8= & 56 \\ \hline & 245225267436\ \textit{Answer.} \end{array}$$

Division may also be performed by this method, by using the reciprocal of the divisor as the multiplier.

Example: Divide 53684 by 45678.

Recip. of 45678 = ·00002189238 nearly.

$$\begin{array}{rlr} 53684 & & \\ \cdot 000021892 & & \\ \hline s.\ 75576 & T_s= & 1427932944 \\ \hline d.\ 31792 & T_d= & 252682816 \\ & & \hline & & 1\cdot 175250128\ \textit{Answer.} \end{array}$$

53684 ÷ 45678 = 1·175272...

The result thus shown is true to four places of decimals; but, by using the reciprocal to six places of significant figures, a greater degree of exactness may be attained. The quotient of the last example would thereby be made true to six places.

By means of these tables we may also obtain the squares of all integer numbers up to the base 100,000, by merely multiplying the tabular values by 4.

I submit, then, that calculations by this method may be advantageously compared with those made both in the ordinary way and by the aid of logarithms. For although, when using the tables I have been describing, it is necessary first to take the sum and difference of the factors—a process which does not occur in logarithmic computations—yet we are saved the more troublesome reverse finding to obtain the natural number corresponding to the sum of the logarithms previously found, as the difference of the tabular values by this method gives the exact product of the factors. It is therefore, I think, apparent that it has considerable advantage over and is more expeditious than the logarithmic method; and it is important to observe that it possesses the further advantage of perfect precision in its results.

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* Practically, these figures need not be set down; they are here only inserted for perspicuity.

The following examples are presented comparatively, in order that the meeting may judge of the merits of the new method:—

To find the product of 53·684 × 45·678.

By the Tables.	By the Common Method.	By Logarithms.
s. 99362, T_s=2468201761	53·684	λ2·7298449
d. 8006, T_d= 16024009	45·678	λ2·6597071
2452·177752	429472	5·3895520
	375788	5·3895382=2452·1
	322104	138
	268420	124= 7
	214736	140
	2452·177752	124= 7
		160
		159= 9
		2452·1779000

Abstract of a Paper on the Progressive Rates of Mortality, as occurring in all Ages, and on certain Deviations; also on Dr. Buchanan's "Physiological Law of Mortality." By JOHN REID, *Surgeon, Glasgow, Author of "The Philosophy of Death."*

[Read before the Institute of Actuaries, 28th January, 1856, and ordered by the Council to be printed in abstract.]

THE progressive rates of mortality, from birth to extreme old age, in a given number of any population, are known from the proportionate numbers who die at the different ages. This is the only mode of estimating correctly the expectation or conceived "*security*" of life at any given age—viz., by showing the chance which an individual has of living a certain number of years or shorter periods.

There is no general law of mortality arising from the physiological functions of the animal system, excepting that resulting from the limit to structural endurance: an organ may be destroyed in the first week of life, from some excited disease, which, apart from that, might have remained entire, say to the age of one hundred years. Disease is only an accidental occurrence to an individual; for even the fact of 3,900 children dying the first five years, out of 10,000 born, only shows that all those children have